

DYNAMICS OF BOUNDED CONFIDENCE OPINION IN HETEROGENEOUS SOCIAL NETWORKS: CONCORD AGAINST PARTIAL ANTAGONISM

EVGUENII KURMYSHEV, HÉCTOR A. JUÁREZ, AND RICARDO A. GONZÁLEZ-SILVA

ABSTRACT. Bounded confidence models of opinion dynamics in social networks have been actively studied in recent years, in particular, opinion formation and extremism propagation along with other aspects of social dynamics. In this work, after an analysis of limitations of the Deffuant-Weisbuch (DW) bounded confidence, relative agreement model, we propose the mixed model that takes into account two psychological types of individuals. Concord agents (C-agents) are friendly people; they interact in a way that their opinions get closer always. Agents of the other psychological type show partial antagonism in their interaction (PA-agents). Opinion dynamics in heterogeneous social groups, consisting of agents of the two types, was studied on different social networks: Erdos-Renyi random graphs, small-world networks and complete graphs. Limit cases of the mixed model, pure C- and PA-societies, were also studied. We found that group opinion formation is, qualitatively, almost independent of the topology of networks used in this work. Opinion fragmentation, polarization and consensus are observed in the mixed model at different proportions of PA- and C-agents, depending on the value of initial opinion tolerance of agents. As for the opinion formation and arising of “dissidents”, the opinion dynamics of the C-agents society was found to be similar to that of the DW model, except for the rate of opinion convergence. Nevertheless, mixed societies showed dynamics and bifurcation patterns notably different to those of the DW model. The influence of biased initial conditions over opinion formation in heterogeneous social groups was also studied versus the initial value of opinion uncertainty, varying the proportion of the PA- to C-agents. Bifurcation diagrams showed impressive evolution of collective opinion, in particular, radical change of left to right consensus or vice versa at an opinion uncertainty value equal to 0.7 in the model with the PA/C mixture of population near 50/50.

1. INTRODUCTION

Detailed behavior of every human being is the result of complex physiological and psychological processes that are not well known yet. No one knows precisely neither the dynamics of a single individual nor the way humans interact to each other. Therefore, any modeling of social dynamics inevitably involves a significant simplification of a real problem. In modeling of social processes, macroscopic phenomena are mainly due to the nontrivial collective effects resulting from the interaction of a large number of “simple” elements of a social network, rather than from a complex behavior of single entities. The way to obtain useful results from

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this kind of modeling is to keep a balance between the complexity of individual behavior and the complexity of a social network.

The search for agreements and the reaching of consensus are important aspects of social group dynamics, because they make the position of a group stronger and enhance its impact onto society. Although the concept of opinion is not simple to define formally (it can be quantified as a degree of satisfaction, desire or preference), we see that opinion dynamics is an evolution problem, and therefore, it can be considered as a dynamical system. Examples are the evolution of voting preferences and variations of market demand for products or trademarks of competitors, among others.

Many of the models recently proposed in social dynamics use techniques of statistical physics [Wei71, CS73, Gal02, Lat81, SWS00, DNAW00] (for a detailed review of the topic and the state of the art see [CFL07] and references in it). The underlying concept of these models is a transition rate (or a probability of transition) between different states of a social system, and opinion dynamics is considered in terms of order-disorder transitions.

Agent-based modeling on networks is another rapidly growing, deterministic approach to social dynamics. Models of this kind study collective effects resulting from the interaction of a number of “simple” agents in a social network [Sch03]. Members of social groups are considered to be adaptive rather than rational agents, and no individual strategy is implied except a common rule of opinion updating. Bounded confidence, relative agreement (BC/RA) models of opinion dynamics are an important example of agent-based modeling [AD04, DAWF02, Def06, DNAW00, HK02, Lor07].

Models of opinion dynamics are usually composed of the following basic elements: opinion space, updating rule, updating dynamics and social network.

Opinion Space Let S be an opinion space, such that every opinion x_i of an agent i is in this space, $x_i \in S$. Two kinds of models are distinguished: models that use a discrete opinion space, and others that consider a continuous opinion space. In the first case, an opinion of an agent i is usually a discrete-valued vector in d -Dim space, $x_i \in S \subset \{0, 1\}^d$, that represents the agents opinion over d subjects (topics). It should be noted that almost all computer simulations have been made for $d = 1$; in this case an agent has to choose between two options, $\{0, 1\}$ or $\{-1, 1\}$. Voter model, majority rule and Ising spin models are examples supported by a discrete opinion space [CFL07]. A unifying frame to incorporate all discrete opinion dynamics models was proposed in [Gal05].

There are situations, as the political orientation of a person, in which an agent preference changes smoothly within a range of possibilities, let us say from the extreme left to the extreme right. These situations are usually suitable to be treated within a continuous opinion space, $x_i \in S \subset \mathbb{R}^d$. In practice, the bounded 1-Dim space is used, $x_i \in S \subset [0, 1]^d$ and $d = 1$ [Lor07].

Updating rule All models assume that the change of opinion of an agent depends on opinions of agents related to him. The way to establish the relationship between agents in a social network is commented later on. It depends on a particular social network. When the relationship between agents is established, an agent takes into account the opinions of other agents related to him if and only if those opinions are close to that of a given person. Closeness is usually defined by means of a threshold that varies from a model to another. Some models assume that two

agents i and j interact with each other when $\|x_i - x_j\| < \epsilon$, where ϵ is a parameter of a model. Nevertheless, in other models a new variable $u \in [0, 1]$, called opinion uncertainty or tolerance, is defined. Then, the closeness of interacting agents depends on ϵ and u . In both cases, the updating rule assigns a new value to the opinion of a given agent j . This value depends on the value of its previous opinion and on the opinions of other agents close to the person.

Updating dynamics Once the connections of an agent with others are established, then we have to define the way they interact. It can be pair or group interaction. In the case of pair interaction, the latter is usually unidirectional. Among pairs of connected agents, (i, j) , one agent is considered to be passive, say j , and the other to be active for a given time step. So, for this time step (time unit of updating) every passive agent j updates his opinion as a function of the opinion of active agent i , not vice versa. In the case of group interaction, the opinion of passive (receptive) agent j is updated in function of the average opinion of the agents connected to him. Once the updating of the opinion of a certain number of agents has been carried out, we say that the one iteration or the time step of the model was done. The number of agents updating their opinions in one time step and the number of iterations depend on the model.

Social Network Relationships between agents are usually described by means of a network. A social network consists of a number of agents N , each represented by a vertex (node), and every pair of nodes of interacting agents is connected by an edge (link). The networks commonly used in computer simulation of social networks are the complete network, the uncorrelated random graph proposed by Erdős-Rényi, the small-world network model by Watts and Strogatz, the growing networks (complex heterogeneous networks) by Barabási and Albert, grids and real networks.

Most of the analysis of opinion dynamics focuses on a steady state opinion formation on a static or evolving social network [InKKB09]. The interest is in the study of possible opinion fragmentation, polarization or consensus in different social groups. The formation of steady state opinion clusters is interpreted as a locally ordered state of social groups in a society, in which the agents achieve a local consensus. In these groups, agents share ideas and common values, while a disordered state looks like a fragmented or anarchic society, in which it is impossible to reach agreements.

In opinion dynamics, in particular in BC models, an interaction between agents is usually determined in the manner that the opinion of a passive agent tends to that of the active one. That is the case of pair interaction in the original DW model [DAWF02]. In the Hegselman-Krause model, an agent adopts an average opinion of the nearest neighbors [HK02]. In these models, opinions of interacting agents get closer to each other, and opposition (repulsive interaction) is not considered. Real life interaction between persons is repulsive-attractive usually. So, a number of mechanisms for repulsive interaction in BC models have been proposed and studied recently.

In [JA05], authors use a simple two-threshold, $U < T$, interpretation of the Social Judgment Theory (SJT) in 1D attitude (opinion) space. The decision on attractive or repulsive interaction is made according to the distance between the opinions of a randomly selected pair of agents, one of which is considered to be the passive one. When the distance d_{12} between attitudes of a randomly selected

pair of agents is less than U , the interaction is attractive; when $U < d_{12} < T$, then the attitude of the passive agent does not change; repulsion of opinions takes place when $d_{12} > T$. Thresholds U and T are free parameters of the model that are used for all the population in experiments. Neither opinion uncertainty nor relative agreement are used in this approach.

In papers [HDJ08, HD08] authors also refer to the SJT and propose 2D space of attitudes a_1 and a_2 . The second attitude a_2 is used as an indicator for triggering an attractive to a repulsive interaction in a_1 . That is the case of the asymmetric use of attitudes. The only function of a_2 is to be a trigger to change an attractive into a repulsive interaction in a_1 , and there is no direct influence of a_1 on a_2 ; there is no triggering in a_2 . The updating rule considers three different criteria based on the distances between the two attitudes of a pair of agents. One may expect that there has to be a direct mechanism of repulsive interaction for every given attitude, but not only via an auxiliary attitude which is taken to be equally valid for all the population. Threshold $U = u_1 = u_2$ used in these works can hardly be interpreted as an uncertainty in the opinion of agents. So, neither opinion uncertainty nor relative agreement are used in this model, as it should be in BC/RA models.

One of the most recent works [VPT10] considers the role of mass media and repulsive interactions in a BC continuous-opinion dynamics. Authors introduce repulsive interaction in pairs of agents by random assigning positive and negative weights ($w_{ij} = w_{ji} = \pm 1$) to the links of a complete graph (social network). Then a simple modification in the updating rule of the work [DNAW00] is used. Negative weight $w_{ij} = -1$ on the link between agents i and j causes repulsive interaction between agents, while positive weight $w_{ij} = +1$ represents attractive interaction. So, when weights are assigned to all links of a network, the agents are divided into “enemies” and “friends” for further opinion evolution, because no links or weights are changed. This model takes into account the bound of confidence ϵ , but it does not use a relative agreement and opinion uncertainty. Another spin glass model that takes into account this feature of “enemies” and “friends” was developed in [Gal96]. A review of some Galam models that takes into account heterogeneous beliefs, inflexible and contrarian effects can be found in [Gal08].

Others than BC approaches use repulsive interaction too. As an example, we refer to the Axelrod models of social influence with cultural repulsion [RM10], and to a continuous opinion dynamics model based on the principle of meta-contrast or Self Categorization Theory [Sal06].

In our study of opinion dynamics, we focus basically on social groups of adaptive rather than rational agents, no individual strategy is implied. We study the evolution of opinion of individuals, looking for the clustering of opinions. A self-consistent mechanism of the repulsive-attractive interaction in the frame of BC/RA continuous opinion model is proposed and studied. Randomness is a necessary feature of social interaction because both an individual attitude and the influence of social environment, are easily altered in time and space in a little predictable manner for an individual. Since bounded confidence models are deterministic, stochastic features of opinion dynamics (social “temperature”) are simulated in this work by means of seeding random initial distributions in opinion and tolerance, and personal links in social networks.

The work is organized as follows. In Section 2.1, we briefly describe and analyze the popular DW model of bounded confidence opinion dynamics. In Section 2.2 we

propose a new model of a mixed society that consists of PA- and C-agents. Results of computer simulation of the mixed PA/C-model on different social networks are presented in Section 3. Finally, conclusion and discussion are given in Section 4.

2. MODELS

2.1. Bounded Confidence Models. The *Deffuant-Weisbuch* model, proposed in [DNAW00], uses interval $S = [-1, 1]$ as a continuous opinion space, so the opinion of agent i is $x_i \in [-1, 1]$. In addition, each agent i is characterized by his opinion uncertainty $u_i \in (0, 1]$. An *opinion segment* $s_i = [x_i - u_i, x_i + u_i]$ is assigned to each agent i . For two agents, i and j , their opinion segments overlap if and only if $h_{ij} = \min(x_i + u_i, x_j + u_j) - \max(x_i - u_i, x_j - u_j) > 0$, where h_{ij} is called an *opinion overlap*. If the overlap is strictly positive, then the *relative agreement of agent i with agent j* is defined by the following equation:

$$(1) \quad \frac{h_{ij} - (2u_i - h_{ij})}{2u_i} = \frac{h_{ij}}{u_i} - 1.$$

Opinion dynamics in a network is simulated as follows. Given a social network, a number of edges of the network (pairs of connected agents), N , is selected randomly, so that N is usually equal to the number of agents in the network. In each pair, one of the interacting agents, say agent j , is sampled randomly to be a *passive (receptive) agent*, while the other one, say agent i , is considered to be the *active agent*. If $h_{ij} > u_i$, then the opinion and uncertainty of the passive agent j are updated according to the following rule:

$$(2) \quad \begin{aligned} x_j &= x_j + \mu \left(\frac{h_{ij}}{u_i} - 1 \right) (x_i - x_j) \\ u_j &= u_j + \mu \left(\frac{h_{ij}}{u_i} - 1 \right) (u_i - u_j) \end{aligned}$$

where μ is a *parameter of convergence* such that $\mu \in [0, \frac{1}{2}]$. If $h_{ij} \leq u_i$, there is no change in the opinion and uncertainty of agent j .

As one can see, the bounded confidence relative agreement DW model has the following distinctive features. During the interaction, agents change both their opinions and uncertainties. Agents with low uncertainty (high confidence) tend to be more influential than others. Equation (1) shows that relative agreement is linear in h_{ij}/u_i . Moreover, the condition $h_{ij} > u_i$, imposed in [DAWF02], implies that the updating factor, Eq. (1) is always positive. As a consequence, both the opinion and uncertainty of passive agent get closer to those of the active agent, Eq. (2). In other words, if there is interaction between two agents, then the active agent convinces the passive one, and there are no possibilities of disagreement. The same initial value of uncertainty U is used for almost the whole population, adding a few agents with small values of uncertainty [AD04].

2.2. Mixed PA/C-Model. Social groups are constituted of persons of different psychological types. In order to capture this important feature of social organization we propose a model of opinion in the mixed PA/C society. The PA/C-model involves agents of two psychological types, PA- and C-agents. The basic elements of the model, the opinion space, uncertainty and overlap of opinion intervals are treated in the same way as in the DW model. In the PA/C-model the opinion and

uncertainty of passive (receptive) agent j are changed when $0 \leq h_{i,j} \leq u_i$, in contrast to that of the DW model which excludes an interaction of agents at $h_{i,j} < u_i$. Agents of the two types differ each other in the way they update their opinion in pair interaction.

The updating rule for the opinion and uncertainty of a passive C-agent j is as follows:

$$(3) \quad \begin{aligned} x_j &= x_j + \mu \left(\frac{h_{ij}}{u_i} \right) (x_i - x_j) \\ u_j &= u_j + \mu \left(\frac{h_{ij}}{u_i} \right) (u_i - u_j). \end{aligned}$$

where active agent i can be PA- or C-agent. Because the uncertainty of opinion is a qualitative rather than quantitative variable, we define a relative agreement for C-agents as h_{ij}/u_i , instead of $(h_{ij}/u_i - 1)$ in the DW model. When $h_{ij} < 0$, opinion segments do not overlap and therefore, there is no modification in the opinion and uncertainty. The interaction of passive C-agents is always attractive in the opinion space, similar to that in the original DW model. This behavior gives the name to the psychological type, concord agents. There is no disagreement between interacting agents.

Agents with partial antagonism (PA-agents) represent the other psychological type. Note that in the original BC/RA DW model, the interaction between agents takes place if and only if the overlap of opinion segments of connected agents is sufficiently large, $h_{ij} > u_i$ [DNAW00]. As a consequence of the imposed restriction, the interaction between agents is always attractive, and the DW model does not allow any disagreement. Analyzing the interaction at smaller overlaps of opinion segments, $0 < h_{ij} < u_i$, we see that is formally repulsive. Nevertheless, the repulsive regime is not considered in the DW model. In addition, at this regime the updating factor is mathematically symmetric to the attractive one (see Figure 1). Real life interaction between persons is usually repulsive-attractive. For that reason we modify the DW model, breaking the symmetry of the updating factor in such a way that opinions of two interacting agents can diverge, but not as strong as it could be in the DW model when the overlap of opinion segments is less than an opinion uncertainty of the active agent. The new updating rule for the opinion and uncertainty of a passive (receptive) PA-agent j is as follows:

$$(4) \quad \begin{aligned} x_j &= x_j + \mu_1 \left(\frac{h_{ij}}{2u_i} \right) \left(\frac{h_{ij}}{u_i} - 1 \right) (x_i - x_j) \\ u_j &= u_j + \mu_2 \left(\frac{h_{ij}}{2u_i} \right) \left(\frac{h_{ij}}{u_i} - 1 \right) (u_i - u_j). \end{aligned}$$

where active agent i can be PA- or C-type. In contrast to the DW model, the scaled relative agreement $(h_{ij}/2u_i)(h_{ij}/u_i - 1)$ is used for PA-agents. The scaling factor $(h_{ij}/2u_i)$ decreases repulsion of opinions since $(h_{ij}/2u_i)(h_{ij}/u_i - 1) \in [-0.125, 1]$ for PA-agents, and $(h_{ij}/u_i - 1) \in [-1, 1]$ (see Figure 1). In addition, we relax the restriction $h_{ij} > u_i$ of the DW model to the condition $h_{ij} > 0$, allowing negative values for the relative agreement $(h_{ij}/2u_i)(h_{ij}/u_i - 1)$ when $0 < h_{ij} < u_i$. There is no modification of opinions and uncertainties when $h_{ij} < 0$. Two different convergent parameters μ_1 and μ_2 are used, because the dynamics of opinion is qualitatively similar to the dynamics of uncertainty, but the updating rates of these variables can be certainly different.

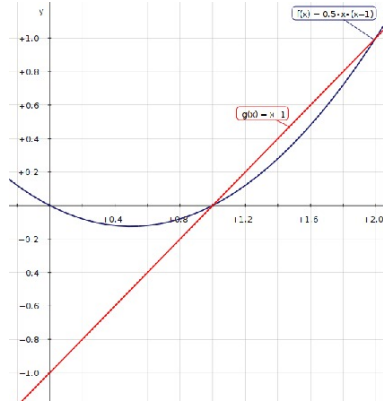


FIGURE 1. The updating factor for PA-type agents.

The PA/C-model considers a society of N individuals of two different psychological types: a fraction p of agents are of the C-type, while the rest of population are PA-agents. The society consists of two sub-populations, M_C and M_{PA} of sizes pN and $(1 - p)N$ respectively, being $0 \leq p \leq 1$ a parameter of the model. When an active agent i interacts with a passive one j , agent j changes its opinion and uncertainty following the rules defined for C-agents if $j \in M_C$ or those defined for PA-agents if $j \in M_{PA}$.

Varying the value of p from 0 to 1, we model mixed societies with populations ranging through C- and PA-type societies. In order to get a more realistic simulation, we use heterogeneous initial conditions instead of the homogeneous distribution in initial opinion that was used in [DNAW00, Def06]. Homogeneous initial opinion in a social network is obtained by drawing individual opinions from the interval $[-1, 1]$ with the uniform probability distribution. In this work, we define a parameter $0 \leq \ell \leq 1$ that describes “political preference” of agents. The initial opinions of ℓN agents are drawn from the interval $[-1, 0]$ with the uniform probability distribution (these agents are considered to be left oriented). Opinions of the rest of population, $(1 - \ell)N$ right oriented agents, are drawn from the interval $[0, 1]$ with the uniform probability distribution also. When $\ell = 0.5$, we have homogeneous initial conditions. So, the new model is doubly heterogeneous:

- Parameter p regulates the psychological composition of the society
- Parameter ℓ changes the initial preference of agents, so we can model right (or left) oriented societies.

These two characteristics result in a rich dynamics of the model.

3. COMPUTER SIMULATION RESULTS

Computer simulations of the model were carried out in a way slightly different to that in [DNAW00, Def06]. The state variables of a social network are the opinion x_i and the uncertainty u_i of individuals. Uniform probability distribution is used to choose the initial uncertainty of each agent from the interval $[U - 0.2, U + 0.2]$, where U is the average initial opinion uncertainty in the network. This condition differs of that in [DNAW00, Def06], where all agents use the same initial value

U for uncertainty. U is a parameter of the model, and we vary it in the interval $0.3 \leq U \leq 1.2$.

At each time step, we choose randomly N pairs of coupled agents, edges of a social network. Then, one agent of each pair is selected at random to be the active agent i , while the other is considered to be the passive one, j . The value of the overlap h_{ij} is computed, and we update variables of the passive agent if and only if $h_{ij} \geq 0$. As mentioned above, the composition of heterogeneous society is regulated with the parameter $0 \leq p \leq 1$, so that $M_C = pN$ and $M_{PA} = (1 - p)N$ is the number of C- and PA-agents, respectively. When an active agent i interacts with a passive one j , agent j changes its attitude according to the C-type rules if $j \in M_C$, or the PA-type rules if $j \in M_{PA}$.

The value of the parameter U runs through 0.3 to 1.2 at a step of 0.01. At each value of U , we execute 350 iterations (time steps). The results shown in the following figures represent an average over 50 simulations for each value of U . Each simulation starts with a new seed of initial conditions in opinion distribution and uncertainty.

Three kinds of networks are used in our experiments. The first is the complete network with 400 agents. The second one is a random network of 1000 agents, such that the degree d_i of each vertex i satisfies $30 \leq d_i \leq 40$, and the distribution of the degrees over the network is uniform. The third network studied in this work is a Watts-Strogatz small world network of 1000 agents, with $k = 20$ and $\beta = 0.25$.

First of all, we are interested in the study of homogeneous societies with primitive democracy. So, neither hubs nor leaders are considered and the initial opinion of each agent is a number randomly drawn from interval $[-1, 1]$, using uniform probability distribution, $\ell = 0.5$. Because we are basically interested in the steady state of opinion dynamics, we plot the density of asymptotic opinions as a function of parameter U . In the following figures, axis x is opinion, and the vertical axis is the number of agents that share an opinion at a given value of U . We observed that all bifurcation patterns of opinion distributions obtained in this work are qualitatively similar for the three social networks under consideration. So, here after, we show only the results obtained for a Small World Network.

Figure 2 shows bifurcation patterns (an average over 50 simulations) of group opinions for different values of p , ranging from $p = 0$ to $p = 1$, and uniform distribution of initial opinions of agents in the interval $[-1, 1]$. Uniform initial distribution of uncertainties in the interval $[U - 0.2, U + 0.2]$ is used for each value of U .

For $p = 0$, the society is composed of PA-agents. Bifurcation of opinion distribution as a function of parameter U is shown in Figure 2a in the interval $0.3 < U < 1.2$. Four opinion clusters are observed for $U < 0.4$, although they are not well separated. For $0.4 \leq U \leq 0.55$ there are three well defined opinion groups. In the interval $0.55 \leq U \leq 1$, there are two opinion groups. Note that the two opinions get more distant from each other as U increases from 0.55 to 1; in other words, as the initial opinion uncertainty (tolerance) of agents gets larger, the society polarizes in two almost extreme opinion groups. However, at $U = 1$ there is another phase transition, and the two opinions collapse into a unique opinion, consensus. So, three bifurcation points are observed at $U = 0.4, 0.55, 1$.

For $p = 1$, the society is composed of C-agents. Bifurcation of opinion distribution is shown in Figure 2h. Note that for $U \leq 0.4$, the C-society is divided in

three groups of agents, two of them with opposite opinions. For $U > 0.4$ there is a unique opinion, consensus, being $U = 0.4$ a bifurcation point of the system. Results of simulations show that the opinion dynamics in the C-model is qualitatively independent of the networks used.

At intermediate values of p , a series of transformations of the bifurcation pattern is observed (see Fig. 2, b through g). For $0 < p < 0.25$ and $U > 0.45$, the bifurcation of the two opinion clusters goes to the consensus through a collapse (see Fig. 2 (a,b and c)), in contrast to the soft convergence of the two opinion clusters into the consensus for $0.25 < p < 0.9$ (see Fig. 2, d through g). Fig. 2 (e,f and g) shows that a mixed society reaches a kind of consensus in the interval $0.4 < U < 0.5$, which becomes unstable for $U > 0.5$, splitting into two group opinions. Then, the two opinions merge softly into the consensus for $U > 1$. For $p \approx 1$, a mixed society reaches a consensus at $U > 0.35$.

Another interesting feature of opinion dynamics in a mixed society is observed in the series of Figs. 2 (e, f, g). Figure 2g shows opinion dynamics of the C-population diluted with 10% of PA-agents. The bifurcation pattern in Figure 2g is similar to that in Fig. 2h for the pure C-society, in the interval of values $0.3 < U < 0.5$. Nevertheless, the consensus breaks out (splits) into two opposite opinions ($\langle u \rangle = +0.6, -0.6$) at $U = 0.6$. Then, those two opinions get closer as U increases, merging into the consensus at $U = 0.85$, once again. Similar behavior of opinion is observed in Figs. 2 (e,f). The important thing is that a relatively small fraction of PA-agents changes opinion dynamics significantly.

Opinion dynamics in a mixed society becomes even more interesting when we switch on non-uniformity in the initial opinion distribution. Let $\ell \in [0, 1]$. We randomly select ℓN agents and assign them initial opinions randomly drawn from the interval $[-1, 0]$ with uniform probability distribution; those are left oriented agents. For the rest of agents, $(1 - \ell)N$, we draw their initial opinions from the interval $[0, 1]$ with uniform probability; they are right oriented. So, the parameter ℓ controls the initial amount of individuals that share the same orientation in their opinions.

We run a number of simulations for different values of p and ℓ . Results of simulation have shown that for a given value of p , the graphs of opinion corresponding to ℓ and $1 - \ell$ are symmetric, as expected. So, we show results only for the right oriented societies ($\ell \in [0, 0.5]$). Figure 3 shows results of computer simulation for $p = 0$ (a society of PA agents) and different values of ℓ . The bifurcation pattern for a totally right oriented society, $\ell = 0$, is shown in Fig. 3a. This pattern is similar to the picture in Fig 2a that was slightly modified and scaled to the interval of opinions $[0, 1]$. We see the formation of the unique opinion cluster centered at $\langle x \rangle = 0.5$ for $0.55 \leq U$. However, there is a small cluster of left oriented agents that do not approve the “main” opinion. As ℓ increases, this group of “dissidents” also grows, and the society gets polarized, as shown in Figures 3b and 3c. In these figures, for $U > 0.55$ there are two clusters of agents with almost opposite opinions, cluster of right oriented agents being greater than the other. The size of the small cluster increases as ℓ increases. As U increases, opinions of the two groups diverge tending to the extremes. The two extreme opinions collapse into the consensus at $U = 1$, for $\ell > 0.4$. For $\ell = 0.5$, we come back to the PA-model with the uniform initial distribution of opinions (compare Figure 3d with Figure 2a).

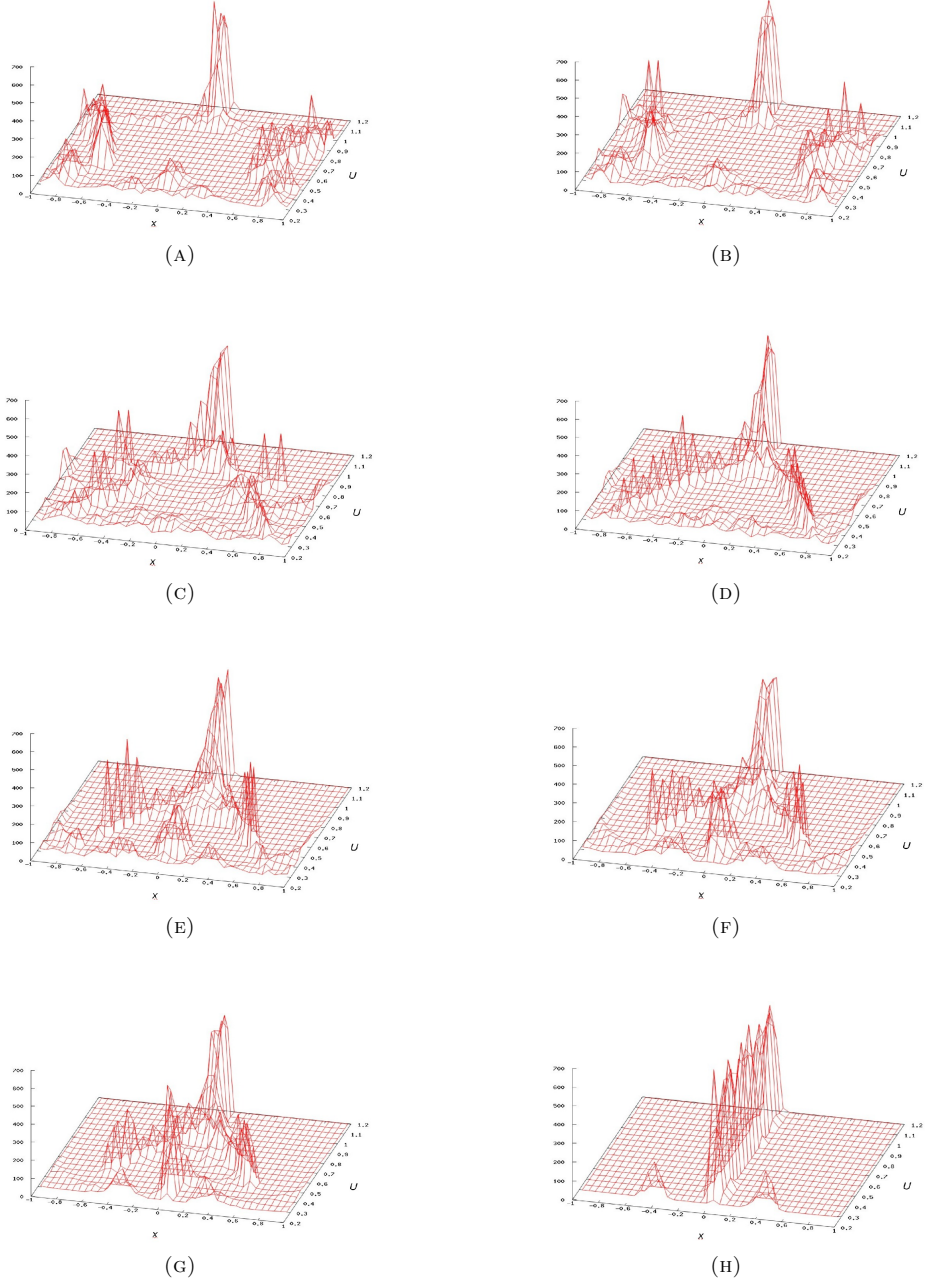


FIGURE 2. Bifurcation of opinion distribution in the mixed model for different values of p . Initial opinions were drawn from the interval $[-1, 1]$, (a) $p = 0.0$, (b) $p = 0.1$, (c) $p = 0.25$, (d) $p = 0.4$, (e) $p = 0.6$, (f) $p = 0.75$, (g) $p = 0.9$ and (h) $p = 1$.

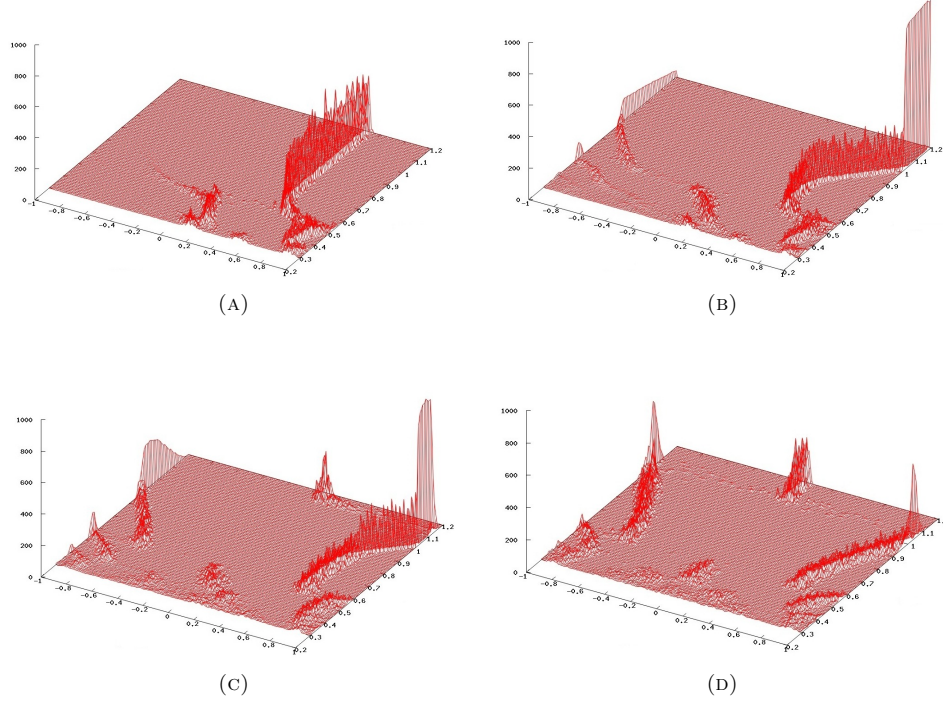


FIGURE 3. Bifurcation of opinion distribution as a function of U in the mixed model for $p = 0$ and (a) $\ell = 0.0$, (b) $\ell = 0.2$, (c) $\ell = 0.4$, (d) $\ell = 0.5$.

For a fraction of C-agents $p < 0.35$, computer simulation showed that the bifurcation pattern of opinion dynamics does change little compared to this of Figure 3. So, in the societies composed mainly of PA agents, opinion dynamics remains basically the same.

Significant changes were observed in opinion dynamics for $p > 0.35$. Figure 4 shows bifurcation patterns for $p = 0.5$, when one half of agents are PA-agents, and the other half are C-agents. In this case the opinion dynamics of the society is strongly influenced by the C-component of the population. For $\ell = 0$ the only one, right-oriented opinion dominates in the society; at the very beginning of the interval of uncertainties $0.3 < U < 1.2$, we observe two small groups of agents with different opinions to the majority of the population. These small groups disappear at $U > 0.4$. When the fraction of left-oriented agents changes in the interval $0 < \ell < 0.45$, a steady state opinion dynamics reveals a little expected behavior of the society (Figures 4b and 4c). For $0.45 < U < 0.7$ there are two opinion groups, one big group of agents on the right and a small cluster of dissidents on the left. For $0.6 < U < 1$ there is a unique group of opinions, but the dominant public opinion suddenly changes from the right to the left at $U = 0.7$. Thus, at $U = 0.7$ the public opinion suffers a phase transition from right to left oriented one. This is a surprising result, showing that the steady state opinion of a society can be very sensitive to the value of tolerance U . For $1 < U$, the whole population endorses

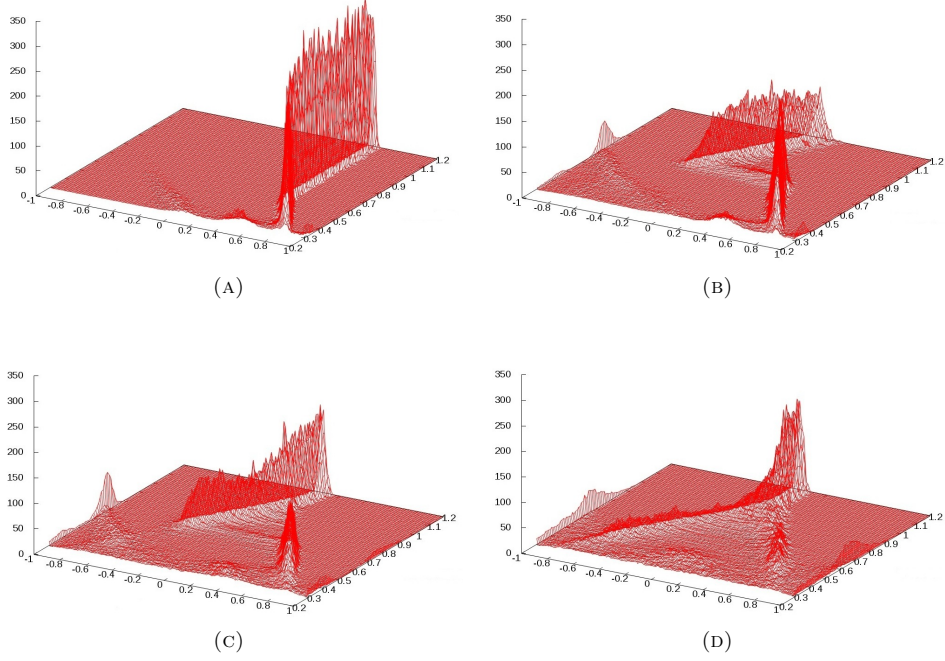


FIGURE 4. Bifurcation of opinion distribution as a function of U in the mixed model at $p = 0.50$ (half of the agents are C-type agents, half are PA-type agents) and (a) $\ell = 0.0$, (b) $\ell = 0.2$, (c) $\ell = 0.4$, (d) $\ell = 0.5$.

an opinion close to 0, and the society becomes center oriented. Figure 4d shows the steady state opinion dynamics when $p = 0.5$ and $\ell = 0.5$, which is similar to Fig. 2d. Two opinion clusters get closer to each other as U increases, and then merge into one cluster, resulting in a λ type pattern. We see that the dynamics shown in Figure 4c, $\ell = 0.4$, and Figure 4d, $\ell = 0.5$, are qualitatively different. Figure 5 shows the transition between the two dynamics. In Figure 5a ($\ell = 0.42$), we observe a dynamics similar to that in Figure 4c, which changes smoothly from Figure 5b ($\ell = 0.45$) to Figure 5c ($\ell = 0.47$), finally resulting in Figure 4d.

4. CONCLUSION AND DISCUSSION

In this work, we proposed a model of opinion formation in a heterogeneous society consisting of agents of two psychological types, with concord and partial antagonism behavior. Clustering of agents in opinion space was studied. To this end, we proposed a bounded confidence, relative agreement model with agents updating their opinions by means of one of the two rules. A concord agent (C-agent) always gets its opinion closer to that of another agent, this differs from the DW model in the way we define relative agreement. PA-agent gets its opinion farther from or closer to that of another agent depending on their relative agreement. In terms of physics, this means a repulsive-attractive potential between agents in the opinion

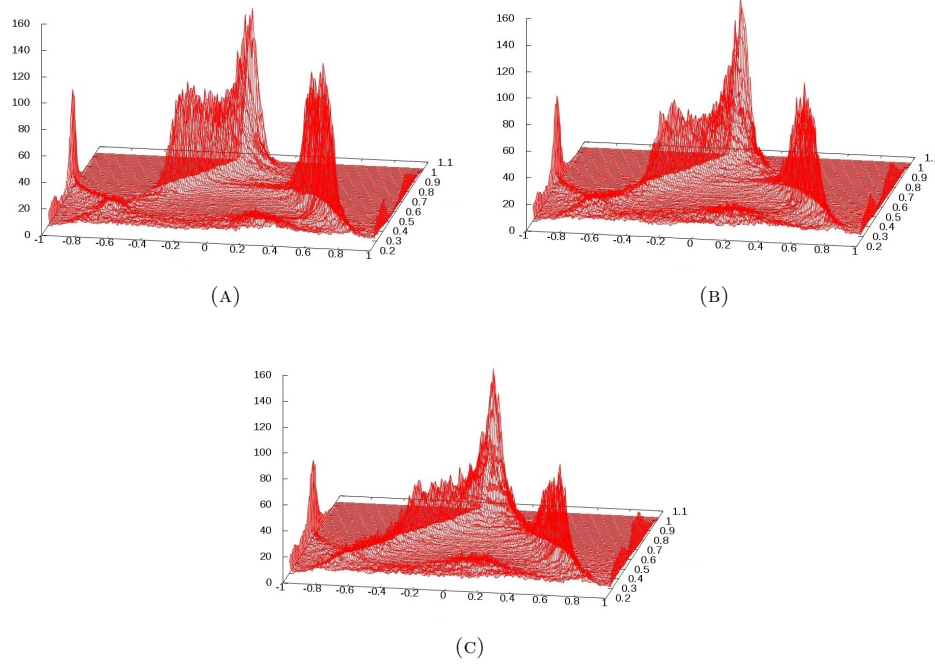


FIGURE 5. Bifurcation of opinion distribution as a function of U in the mixed model at $p = 0.50$ (half of the agents are C-type agents, half are PA-type agents) and (a) $\ell = 0.42$, (b) $\ell = 0.45$, (c) $\ell = 0.47$.

space. Opinion formation in a society of agents of different psychological types was simulated in the mixed model varying the ratio of PA- to C-agents.

In order to study opinion formation in time, pair interaction between agents was used in the model. Varying the initial mean value U of the opinion tolerance of agents, in our model we observed fragmentation, polarization or consensus. Computer simulations show that a steady state opinion is qualitatively little sensitive to the three networks used in the proposed model. In addition, a smaller opinion tolerance (uncertainty) causes a larger fragmentation of opinion. So, a society of close minded persons tends to be partitioned into a large number of small groups of agents with similar opinion. Details of fragmentation of a society in the opinion space depend on the updating rule of opinion and a social network.

The dynamics of opinion formation and bifurcation patterns depend on the value of the parameter p in the mixed model. For example, time convergence to one of the asymptotic opinion states (fragmentation, polarization or consensus) was much faster in the case of C-agents ($p = 1$) than that of the mixed models ($0 \leq p < 1$). As we expected, opinion formation in the C-society was qualitatively similar to that of the DW model, even though the direct comparison of the results is not possible because we used initial conditions for opinion and uncertainty (tolerance) different to those in [DNAW00]. We found that in the mixed model the transition of opinion from one asymptotic state to another is a bifurcation, depending on

the initial mean value of opinion uncertainty of agents. Even though bifurcations are observed in all bounded confidence models studied in [DNAW00, Lor07], these differ from the bifurcations observed in our model, especially for $p < 1$. The main difference is that the DW model shows repetitive interruption of the line of the centrist opinion cluster followed by scaling in the bifurcation diagram (see Fig. 1 in [Lor07]), while branching in our model for $p < 1$ goes from odd to even number of $1 - 2 - 3 - 4 - \dots$ branches without scaling (see Fig. 2). All the lateral branches of the bifurcation diagram in the DW model tend to converge to the central line (centrist opinion) as the initial tolerance U increases, in contrast to our model where lateral branches of fragmented opinion clearly diverge for $p < 1$ as U increases. In addition, an interesting phenomenon was observed in the bifurcation pattern for $p = 0$ and uniform initial distribution of opinion (see Fig. 2a); when initial uncertainty in opinion increases in the interval $0.55 \leq U \leq 1$, two equal groups of agents have opposite opinions that diverge almost to the extremes, -1 and 1 , until they suddenly collapse into the consensus. It looks like the more open-minded social groups tend to separate more from each other before they reach a consensus. Our vague “sociological” explanation of this dynamics is that “clever = open-minded” groups of agents initially tend to emphasize their differences in opinion (idea). However, when they become as “clever = open-minded” as they could understand and accept the opponents idea, they get to the consensus. This bifurcation pattern differs from that of the DW model, in which the two branches of polarized opinion converge gradually into the consensus [DNAW00, Lor07]. In this concern, the sociologists interpretation of the observed dynamics would be very much valuable. A formal mathematical analysis, classification and comparison of bifurcations of all the bounded confidence opinion models should be done.

An important feature of this work is the use of biased initial conditions in opinion besides the uniform distribution of initial opinion in a social group, in contrast to what is usually considered in previous works [DNAW00, Lor07], with the exception of maybe a particular case of a society of open- and close-minded agents [Lor10]. Uniform initial conditions can be interpreted as a state of complete democracy that further evolves to a symmetric fragmentation of a steady state opinion, usually. But how do opinions evolve when a social group initially has two subgroups of different size, and the average opinion of one subgroup is “left-oriented” and that of the other is “right-oriented”? To simulate this situation, we used a piece-wise uniform distribution of initial opinion, varying the ratio of a number of left- to right-oriented agents, including limit cases when one of these subgroups does not exist. The simulation of opinion formation in our mixed society shows an extremely interesting behavior of the society. When nearly half the population of a social group were PA-agents, and the other part were C-agents, $p \approx 0.5$, we varied the ratio of carriers of “left” to “right” ideas, parameter ℓ , from 0 to 1. When ℓ was in the interval $\ell = (0.15, 0.42)$ (the society initially has “right” ideas), we found that at $U = 0.7$, the main branch of a steady state opinion diagram bifurcates drastically from “right” to “left” (the opinion of the majority of the group changes from $+0.5$ to -0.5 , approximately), or vice versa, when $\ell = (0.58, 0.95)$. From a sociological point of view, it is a critical behavior in opinion formation. This “mechanism” can explain unexpected results in a voting process, when the social composition and the initial opinion state of a society have not been assessed correctly. In addition, when the main branch is asymmetric, subgroups of opponents (dissidents) and

centrists are also observed. A tiny dissidents branch that tends to the opposite extreme opinion when the initial uncertainty increases was always observed for any tolerance, at $p = 0.5$. This can provide a mechanism for the formation of extremism.

As it has been shown, the main features of opinion formation as fragmentation, polarization, consensus, centrism and extremism in opinion space emerged naturally in our models. In addition, the Mixed model studied in this work is “doubly” heterogeneous. First, it describes a heterogeneous society with a different ratio of PA- to C-agents. Second, we use heterogeneous initial conditions in opinion and tolerance. Piece-wise homogeneous distributions are used for initial opinion. Also, we assign different initial tolerances to agents within a relatively wide interval of values near the mean value U , so that a society has a variety of agents between close- and open-minded ones. All these characteristics show that the model proposed and studied in this work provide a mechanism by means of which the formation of opinion in different social groups can be simulated and explained.

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(E. Kurmyshev) CENTRO DE INVESTIGACIONES EN ÓPTICA, LOMA DEL BOSQUE 115, COL. LOMAS DEL CAMPESTRE, CP 37150, LEÓN, GUANAJUATO, MÉXICO

Current address, E. Kurmyshev: Centro Universitario de Los Lagos — Universidad de Guadalajara, Enrique Díaz de León 1144, Col. Paseos de la Montaña, CP 47460, Lagos de Moreno, Jalisco, México

E-mail address, E. Kurmyshev: `kev@cio.mx`

E-mail address, E. Kurmyshev: `ekurmyshev@culagos.udg.mx`

(H.A. Juárez and R.A. González-Silva) CENTRO UNIVERSITARIO DE LOS LAGOS — UNIVERSIDAD DE GUADALAJARA, ENRIQUE DÍAZ DE LEÓN 1144, COL. PASEOS DE LA MONTAÑA, CP 47460, LAGOS DE MORENO, JALISCO, MÉXICO

E-mail address, H.A. Juárez: `hjuarez@culagos.udg.mx`

E-mail address, R.A. González-Silva: `rgonzalez@culagos.udg.mx`